Mathematical Statistician and Engineering Applications
ISSN: 2094-0343
2326-9865

Detour Global Domination for Splitting Graph of Path, Cycle, Wheel and Helm graph

C. Jayasekaran1, S.V. Ashwin Prakash2
1Associate Professor, Department of Mathematics,
Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India.
2Research Scholar, Reg. No.: 20113132091001, Department of Mathematics,
Pioneer Kumaraswamy College, Nagercoil - 629003, Tamil Nadu, India.
Affliated to Manonmaniam Sundaranar University, Abishekapatti,
Tirunelveli - 627012,Tamil Nadu, India.
email : jayacpkc@gmail.com1, ashwinprakash00@gmail.com2

Article Info
Page Number: 185 - 194
Publication Issue:
Vol 72 No. 1 (2023)

Article History
***Article Received**: 15 October 2022*
***Revised**: 24 November 2022*
***Accepted**: 18 December 2022*

Abstract

In this paper, we introduced the new concept detour global domination number for splitting graph of Path, Cycle, Wheel and Helm graph. First we recollect the concept of splitting graph of a graph and we produce some results based on the detour global domination number of splitting graph of path graph, cycle graph, wheel graph and helm graph. A set S is called a *detour global dominating set* of G if S is both detour and global dominating set of G. The *detour global domination number* is the minimum cardinality of a detour global dominating set in G.

Keywords: Detour set, Dominating set, Detour Domination, Global Domination, Detour Global Domination, Splitting graphs.

---

1 Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order*$|V|$ and *size*$|E|$ of G are denoted by *p* and *q* respectively. For graph theoretic terminology we refer to West[8]. A vertex v of G is said to be an *extreme vertex* if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by $Ext(G)$. For vertices x and y in a connected graph G, the *detour distance*$D(x, y)$ is the length of a longest $x - y$ path in G[1]. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G. For $S \subseteq V, I_D[S] = \cup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set* [2].

A set $S \subseteq V(G)$ in a graph G is a *dominating set* of G if for every vertex v in V-S, there exists a vertex $u \in S$ such that v is adjacent to u. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G3. The complement $\overline{G}$ of a graph G also has $V(G)$ as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A set $S \subseteq V(G)$ is called a *global dominating set* of G if it is a dominating set of both G and $\overline{G}$6. While studying Detour global domination for splitting graphs of some graphs[5], we felt that the detour global domination number for splitting graph of Path, Cycle, Wheel and Helm can also be studied and that is the motivation for this study.

Definition 1.1 *[4] Let $G = (V, E)$ be a connected graph with atleast two vertices. A set $S \subseteq V(G)$ is said to be a detour global dominating set of G if S is both detour and global dominating set of G. The detour global domination number, denoted by $\bar{\gamma}_d(G)$ is the minimum cardinality of a detour global dominating set of G and the detour global dominating set with cardinality $\bar{\gamma}_d(G)$ is called the $\bar{\gamma}_d$-set of G or $\bar{\gamma}_d(G)$-set.*

Definition 1.2 *[7] The splitting graph $S'(G)$ of G is obtained by adding a new vertex v' corresponding to each vertex v of G such that v' is adjacent to every vertex adjacent to v in G. If n is the order of G, then $2n$ is the order of $S'(G)$. We say that the vertices $v_1, v_2, \cdots, v_n$ are duplicated by $v_{1'}, v_{2'}, \cdots, v_{n'}$.*

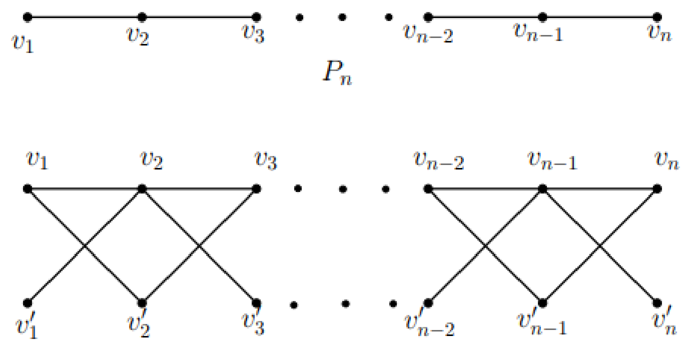
Theorem 1.3 *[4] Every end vertex of G belongs to every detour global dominating set of G.*

2 Splitting graphs of Path, Cycle, Wheel and Helm graph and their Detour Global Domination number

Theorem 2.1 *For any integer $n \geq 2$,* $\bar{\gamma}_d(S'(P_n)) = \begin{cases} n & if \quad n = 2,3 \\ \lceil \frac{n+4}{2} \rceil & if \quad n \geq 4 \end{cases}$

Proof. Let $v_1 v_2 \cdots v_n$ be the path P_n and $v_{1'}, v_{2'}, v_{3'}, \cdots, v_{n'}$ be the corresponding duplicated vertices of $v_1, v_2, \cdots, v_n$ respectively added to obtain $S'(P_n)$ and the graph is shown in Figure 2.1. Then $V(S'(P_n)) = \{v_1, v_2, \cdots, v_n, v_{1'}, v_{2'}, \cdots, v_{n'}\}$ and so $|V(S'(P_n))| = 2n$.

Since $v_{1'}$ and $v_{n'}$ are the only end vertices of $S'(P_n)$, by Theorem 1.3, $v_{1'}$ and $v_{n'}$ are in every detour global dominating set of $S'(P_n)$. Let $W = \{v_{1'}, v_{n'}\}$ be a subset of minimum detour set.



$S'(P_n)$
Figure 2.1

If $n = 2$, then W itself forms a detour global dominating set of $S'(P_2)$ and so $\bar{\gamma}_d(S'(P_2)) = |W| = 2$.
For $n = 3$, W cannot be a detour set. Hence, to obtain a detour set we include $v_{2'}$ with W. Clearly, $I_D[W \cup \{v_{2'}\}] = V(S'(P_3))$. Hence, $W \cup \{v_{2'}\}$ is a minimum detour set. Also, $W \cup \{v_{2'}\}$ dominates every vertices in $S'(P_3)$. Hence, $W \cup \{v_{2'}\}$ is a minimum detour dominating set. Now in $\overline{S'(P_3)}$, $v_{2'}$ dominates $v_2, v_{1'}$ and $v_{3'}$ and $v_{3'}$ dominates every vertices other than v_2. Clearly, $W \cup \{v_{2'}\}$ forms a detour global dominating set of $S'(P_3)$ and so $\bar{\gamma}_d(S'(P_3)) = |W \cup \{v_{2'}\}| = 3 = n$.
Now for $n \geq 4$, let S be a minimum detour global dominating set of $S'(P_n)$. Since, v_1 and v_n are not dominated by any vertex of W, W itself is not a detour global dominating set of $S'(P_n)$. Also, every detour global dominating set of $S'(P_n)$ contains W. Therefore, $W \subseteq S$. Now we need to construct S as a detour global dominating set of $S'(P_n)$ for this we consider the following four cases.
Case (i) $n \equiv 0$ (mod 4)
We consider the splitting graphs $S'(P_{4k})$ where $k = 1,2,3, \cdots$. Here, $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 2(mod\ 4)$ and $N[v_3] = \{v_2, v_3, v_4, v_{2'}, v_{4'}\}$; $N[v_7] = \{v_6, v_7, v_8, v_{6'}, v_{8'}\}$, $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 3(mod\ 4)$. Clearly, $N[v_2] \cup N[v_3] \cup N[v_6] \cup N[v_7] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-2}] \cup N[v_{n-1}] = V(P_n)$. Hence we obtain a dominating set $S = W \cup \{v_2, v_3, v_6, v_7, \cdots, v_x, v_y, \cdots, v_{n-2}, v_{n-1}\}$ of $S'(P_n)$ where $x \equiv 2(mod\ 4)$ and $y \equiv 3(mod\ 4)$. Also, $I_D[v_{1'}, v_{n'}] = V(S'(P_n)$. Hence, S forms a detour set of $S'(P_n)$ and so S is a detour dominating set of $S'(P_n)$. Also, in $\overline{S'(P_n)}$, $v_{1'}$ dominates every vertices other than v_2. Since, $v_{1'}, v_2 \in S$, S is a minimum detour global

dominating set of $S'(P_n)$ and so $\bar{\gamma}_d(S'(P_n)) = |S| = 2 + \frac{n}{2} = \frac{n+4}{2} = \lceil\frac{n+4}{2}\rceil$.

Case (ii) $n \equiv 1(mod\ \ 4)$

We consider the splitting graphs $S'(P_{4k+1})$ where $k = 1,2,3,\cdots$ Here, $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 2(mod\ \ 4)$ and $N[v_3] = \{v_2, v_3, v_4, v_{2'}, v_{4'}\}$; $N[v_7] = \{v_6, v_7, v_8, v_{6'}, v_{8'}\}$, $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 3(mod\ \ 4)$. Also, we include $N[v_n] = \{v_{n-1}, {v_{n-1}}'\}$ Clearly, $N[v_2] \cup N[v_3] \cup N[v_6] \cup N[v_7] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-3}] \cup N[v_{n-2}] \cup N[v_n] = V(P_n)$. Hence we obtain a dominating set $S = W \cup \{v_2, v_3, v_6, v_7, \cdots, v_x, v_y, \cdots, v_{n-3}, v_{n-2}, v_n\}$ of $S'(P_n)$ where $x \equiv 2(mod\ \ 4)$ and $y \equiv 3(mod\ \ 4)$. Also, $I_D[v_{1'}, v_{n'}] = V(S'(P_n)$. Hence, S forms a detour set of $S'(P_n)$ and so S is a detour dominating set of $S'(P_n)$. Also, in $\overline{S'(P_n)}$, $v_{1'}$ dominates every vertices other than v_2. Since, $v_{1'}, v_2 \in S$, S is a minimum detour global dominating set of $S'(P_n)$ and so $\bar{\gamma}_d(S'(P_n)) = |S| = 2 + \lceil\frac{n}{2}\rceil = \lceil\frac{n+4}{2}\rceil$.

Case (iii) $n \equiv 2(mod\ \ 4)$

We consider the splitting graphs $S'(P_{4k+2})$ where $k = 1,2,3,\cdots$ Here, $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 2(mod\ \ 4)$ and $N[v_3] = \{v_2, v_3, v_4, v_{2'}, v_{4'}\}$; $N[v_7] = \{v_6, v_7, v_8, v_{6'}, v_{8'}\}$, $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 3(mod\ \ 4)$ and $N[v_n] = \{v_{n-1}, {v_{n-1}}'\}$ Clearly, $N[v_2] \cup N[v_3] \cup N[v_6] \cup N[v_7] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-4}] \cup N[v_{n-3}] \cup N[v_n] = V(P_n)$. Hence we obtain a dominating set $S = W \cup \{v_2, v_3, v_6, v_7, \cdots, v_x, v_y, \cdots, v_{n-4}, v_{n-3}, v_n\}$ of $S'(P_n)$ where $x \equiv 2(mod\ \ 4)$ and $y \equiv 3(mod\ \ 4)$. Also, $I_D[v_{1'}, v_{n'}] = V(S'(P_n)$. Hence, S forms a detour set of $S'(P_n)$ and so S is a detour dominating set of $S'(P_n)$. Also, in $\overline{S'(P_n)}$, $v_{1'}$ dominates every vertices other than v_2. Since, $v_{1'}, v_2 \in S$, S is a minimum detour global dominating set of $S'(P_n)$ and so $\bar{\gamma}_d(S'(P_n)) = |S| = 2 + \lceil\frac{n}{2}\rceil = \lceil\frac{n+4}{2}\rceil$.

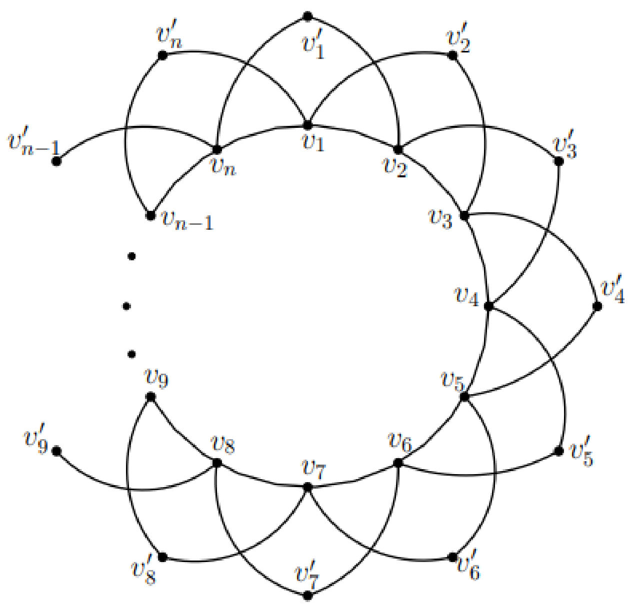
Case (iv) $n \equiv 3(mod\ \ 4)$

We consider the splitting graphs $S'(P_{4k+3})$ where $k = 1,2,3,\cdots$ Here, $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 2(mod\ \ 4)$ and $N[v_3] = \{v_2, v_3, v_4, v_{2'}, v_{4'}\}$; $N[v_7] = \{v_6, v_7, v_8, v_{6'}, v_{8'}\}$, $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 3(mod\ \ 4)$ and $N[v_n] = \{v_{n-1}, {v_{n-1}}'\}$ Clearly, $N[v_2] \cup N[v_3] \cup N[v_6] \cup N[v_7] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-1}] \cup N[v_n] = V(P_n)$. Hence we obtain a dominating set $S = W \cup \{v_2, v_3, v_6, v_7, \cdots, v_x, v_y, \cdots, v_{n-1}, v_n\}$ of $S'(P_n)$ where $x \equiv 2(mod\ \ 4)$ and $y \equiv 3(mod\ \ 4)$. Also, $I_D[v_{1'}, v_{n'}] = V(S'(P_n)$. Hence, S forms a detour set of $S'(P_n)$ and so S is a detour dominating set of $S'(P_n)$. Also, in $\overline{S'(P_n)}$, $v_{1'}$ dominates every vertices other than v_2. Since, $v_{1'}, v_2 \in S$, S is a minimum detour global dominating set of $S'(P_n)$ and so $\bar{\gamma}_d(S'(P_n)) = |S| = 2 + \lceil \frac{n}{2} \rceil = \lceil \frac{n+4}{2} \rceil$.

Thus, $\bar{\gamma}_d(S'(P_n)) = \begin{cases} n & if \quad n = 2,3 \\ \lceil \frac{n+4}{2} \rceil & if \quad n \geq 4 \end{cases}$

Theorem 2.2 *For any integer* $n \geq 4$, $\bar{\gamma}_d(S'(C_n)) = \begin{cases} \lceil \frac{n+1}{2} \rceil & if \quad n \equiv 2(mod 4) \\ \lceil \frac{n}{2} \rceil & otherwise \end{cases}$

Proof. Let $v_1 v_2 \cdots v_n$ be the cycle C_n and $v_{1'}, v_{2'}, v_{3'}, \cdots, v_{n'}$ be the corresponding duplicated vertices of $v_1, v_2, \cdots, v_n$ respectively added to obtain $S'(C_n)$ and the graph is shown in Figure 2.2 . Then $V(S'(C_n)) = \{v_1, v_2, \cdots, v_n, v_{1'}, v_{2'}, \cdots, v_{n'}\}$ and so $|V(S'(C_n))| = 2n$.



$S'(C_n)$
Figure 2.2

Let S be a detour global dominating set of $S'(C_n)$. To construct S as a detour global dominating set of $S'(C_n)$ we consider the following four cases.

Case (i) $n \equiv 0$ (mod 4)

We consider the splitting graphs $S'(C_{4k})$ where $k = 1,2,3,\cdots$. Here, $N[v_1] = \{v_1, v_2, v_n, v_{2'}, v_{n'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{4'}, v_{6'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 1(mod\ \ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$;

$N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 2(mod\ \ 4)$. Clearly, $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-3}] \cup N[v_{n-2}] = V(C_n)$. Hence we obtain a dominating set $S = \{v_1, v_2, v_5, v_6, \cdots, v_x, v_y, \cdots, v_{n-3}, v_{n-2}\}$ of $S'(C_n)$ where $x \equiv 1(mod\ \ 4)$ and $y \equiv 2(mod\ \ 4)$. Also, $I_D[v_1, v_2] = V(S'(C_n)$. Hence, S forms a detour set of $S'(C_n)$ and so S is a detour dominating set of $S'(C_n)$. Also, in $\overline{S'(C_n)}$, v_1 dominates every vertices other than $v_2, v_n, v_{n'}$ and $v_{2'}$; v_2 dominates every vertices other than $v_1, v_3, v_{1'}$ and $v_{3'}$. Clearly, v_1 and v_2 dominate every vertices in $\overline{S'(C_n)}$. Thus, S is a minimum detour global dominating set of $S'(C_n)$ and so $\bar{\gamma}_d(S'(C_n)) = |S| = \lceil\frac{n}{2}\rceil$.

Case (ii) $n \equiv 1(mod\ \ 4)$

We consider the splitting graphs $S'(C_{4k+1})$ where $k = 1,2,3, \cdots$ Here, $N[v_1] = \{v_1, v_2, v_n, v_{2'}, v_{n'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{4'}, v_{6'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 1(mod\ \ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 2(mod\ \ 4)$. Clearly, $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_n] = V(C_n)$. Hence we obtain a dominating set $S = \{v_1, v_2, v_5, v_6, \cdots, v_x, v_y, \cdots, v_n\}$ of $S'(C_n)$ where $x \equiv 1(mod\ \ 4)$ and $y \equiv 2(mod\ \ 4)$. Also, $I_D[v_1, v_2] = V(S'(C_n)$. Hence, S forms a detour set of $S'(C_n)$ and so S is a detour dominating set of $S'(C_n)$. Also, in $\overline{S'(C_n)}$, v_1 dominates every vertices other than $v_2, v_3, v_{2'}$ and $v_{3'}$; v_2 dominates every vertices other than $v_1, v_3, v_{1'}$ and $v_{3'}$. Clearly, v_1 and v_2 dominates every vertices in $\overline{S'(C_n)}$. Thus, S is a minimum detour global dominating set of $S'(C_n)$ and so $\bar{\gamma}_d(S'(C_n)) = |S| = \lceil\frac{n}{2}\rceil$.

Case (iii) $n \equiv 2(mod\ \ 4)$

We consider the splitting graphs $S'(C_{4k+2})$ where $k = 1,2,3, \cdots$ Here, $N[v_1] = \{v_1, v_2, v_n, v_{2'}, v_{n'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{4'}, v_{6'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 1(mod\ \ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 2(mod\ \ 4)$. Clearly, $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-1}] \cup N[v_n] = V(C_n)$. Hence we obtain a dominating set $S = \{v_1, v_2, v_5, v_6, \cdots, v_x, v_y, \cdots, v_{n-1}, v_n\}$ of $S'(C_n)$ where $x \equiv 1(mod\ \ 4)$ and $y \equiv 2(mod\ \ 4)$. Also, $I_D[v_1, v_2] = V(S'(C_n)$. Hence, S forms a detour set of $S'(C_n)$ and so S is a detour dominating set of $S'(C_n)$. Also, in $\overline{S'(C_n)}$, v_1 dominates every vertices other than $v_2, v_3, v_{2'}$ and $v_{3'}$; v_2 dominates every vertices other than $v_1, v_3, v_{1'}$ and $v_{3'}$. Clearly, v_1 and v_2 dominates every vertices in $\overline{S'(C_n)}$. Thus, S is a minimum detour global dominating set of $S'(C_n)$ and so $\bar{\gamma}_d(S'(C_n)) = |S| = \lceil\frac{n+1}{2}\rceil$.

Case (iv) $n \equiv 3(mod\ \ 4)$

We consider the splitting graphs $S'(C_{4k+3})$ where $k = 1,2,3, \cdots$ Here,

$N[v_1] = \{v_1, v_2, v_n, v_{2'}, v_{n'}\}$; $N[v_5] = \{v_4, v_5, v_6, v_{4'}, v_{6'}\}$; $N[v_x] = \{v_{x-1}, v_x, v_{x+1}, {v_{x-1}}', {v_{x+1}}'\}$ where $x \equiv 1(mod\ \ 4)$ and $N[v_2] = \{v_1, v_2, v_3, v_{1'}, v_{3'}\}$; $N[v_6] = \{v_5, v_6, v_7, v_{5'}, v_{7'}\}$; $N[v_y] = \{v_{y-1}, v_y, v_{y+1}, {v_{y-1}}', {v_{y+1}}'\}$ where $y \equiv 2(mod\ \ 4)$. Clearly, $N[v_1] \cup N[v_2] \cup N[v_5] \cup N[v_6] \cup \cdots \cup N[v_x] \cup N[v_y] \cdots \cup N[v_{n-2}] \cup N[v_{n-1}] = V(C_n)$. Hence we obtain a dominating set $S = \{v_1, v_2, v_5, v_6, \cdots, v_x, v_y, \cdots, v_{n-2}, v_{n-1}\}$ of $S'(C_n)$ where $x \equiv 1(mod\ \ 4)$ and $y \equiv 2(mod\ \ 4)$. Also, $I_D[v_1, v_2] = V(S'(C_n)$. Hence, S forms a detour set of $S'(C_n)$ and so S is a detour dominating set of $S'(C_n)$. Also, in $\overline{S'(C_n)}$, v_1 dominates every vertices other than $v_2, v_3, v_{2'}$ and $v_{3'}$; v_2 dominates every vertices other than $v_1, v_3, v_{1'}$ and $v_{3'}$. Clearly, v_1 and v_2 dominates every vertices in $\overline{S'(C_n)}$. Thus, S is a minimum detour global dominating set of $S'(C_n)$ and so $\bar{\gamma}_d(S'(C_n)) = |S| = \lceil\frac{n}{2}\rceil$.

Thus, $\bar{\gamma}_d(S'(C_n)) = \begin{cases} \lceil\frac{n+1}{2}\rceil & if\ \ n \equiv 2(mod4) \\ \lceil\frac{n}{2}\rceil & otherwise \end{cases}$

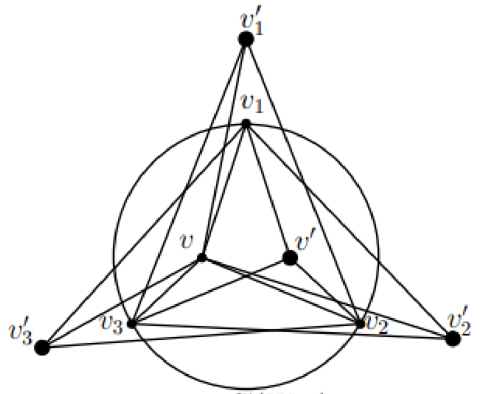
Theorem 2.3 *For the wheel graph* $W_{1,n-1}$*,* $\bar{\gamma}_d(S'(W_{1,n-1})) = 4$*.*

Proof. Let $v_1, v_2, \cdots v_{n-1}$ be the rim vertices of $W_{1,n-1}$ and v be the apex vertex of $W_{1,n-1}$. Let $v_{1'}, v_{2'}, v_{3'} \cdots, {v_{n-1}}'$ be the corresponding duplicated vertices of $v_1, v_2, \cdots, v_{n-1}$ respectively and v' be the duplicated vertex of v which are added to obtain $S'(W_{1,n-1})$. Then $V(S'(W_{1,n-1})) = \{v, v_1, v_2, \cdots, v_{n-1}, v', v_{1'}, v_{2'}, \cdots, {v_{n-1}}'\}$ and $|V(S'(W_{1,n-1}))| = 2n$. Also, $S'(W_{1,n-1})$ has no end vertices and no full vertices.

Here, v is adjacent to every vertices other than v' and v' is adjacent to every $v_i, 1 \le i \le n-1$. Now, to construct a detour global dominating set of $S'(W_{1,n-1})$ we consider the following two cases.

Case (i) $n = 4$

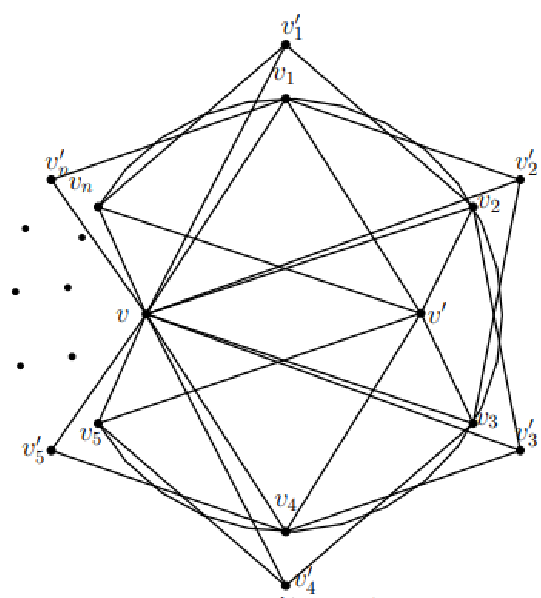
In $\overline{S'(W_{1,3})}$ given in figure 2.3, $N[v_{1'}] = \{v_{1'}, v_{2'}, v_{3'}, v', v_1\}; N[v_{2'}] = \{v_{1'}, v_{2'}, v_{3'}, v', v_2\}; N[v_{3'}] = \{v_{1'}, v_{2'}, v_{3'}, v', v_3\}; N[v_1] = \{v_1, v_{1'}\}; N[v_2] = \{v_2, v_{2'}\}; N[v_3] = \{v_3, v_{3'}\}; N[v] = \{v, v'\}; N[v'] = \{v, v', v_{1'}, v_{2'}, v_{3'}\}$. Here, $S = \{v_{1'}, v_{2'}, v_{3'}\}$ dominates v_1, v_2, v_3 and v' in $\overline{S'(W_{1,3})}$. In order to dominate v we include v' in S. Hence, $S = \{v', v_{1'}, v_{2'}, v_{3'}\}$ is a minimum global dominating set of $S'(W_{1,3})$. Also, S is a minimum dominating set of $S'(W_{1,3})$ and every vertices in $V(S'(W_{1,3}))$ lies on the longest path between v' and $v_{1'}$. Therefore, $S = \{v', v_{1'}, v_{2'}, v_{3'}\}$ is a minimum detour global dominating set of $S'(W_{1,3})$ and so $\bar{\gamma}_d(S'(W_{1,3})) = 4$.



$S'(W_{1,3})$
Figure 2.3

Case (ii) $n \geq 5$

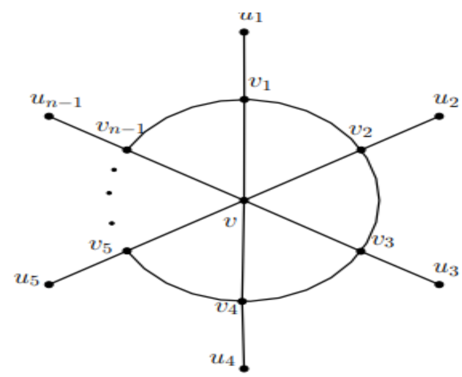
Since, v is adjacent to every vertices other than v' and v' is adjacent to every $v_i, 1 \leq i \leq n-1$. Hence, $S = \{v, v'\}$ is a minimum dominating set of $S'(W_{1,n-1})$. Also, every vertices in $V(S'(W_{1,n-1})) - S$ lies on the longest path between v and v'. Therefore, S is a minimum detour dominating set of $S'(W_{1,n-1})$. In $\overline{S'(W_{1,n-1})}$, v is adjacent to only v' and v' is adjacent to every $v_{i'}, 1 \leq i \leq n-1$ and so S cannot be a detour global dominating set. In $\overline{S'(W_{1,n-1})}$, for $1 \leq i \leq n-1$, $N[v_{i'}] = \{v_{i'}, {v_{i+1}}', \cdots, {v_{i-1}}', v', v_i\}; N[v_i] = \{v_i, v_{i'}\}$. Clearly, $S \cup \{v_{i'}, {v_{i+1}}'\}$ is a dominating set of $\overline{S'(W_{1,n-1})}$. Therefore, for $1 \leq i \leq n-2$, $\{v, v', v_{i'}, {v_{i+1}}'\}$ is a detour global dominating set of $S'(W_{1,n-1})$ and so $\bar{\gamma}_d(S'(W_{1,n-1})) = 4$.
Thus, $W_{1,n-1}$, $\bar{\gamma}_d(S'(W_{1,n-1})) = 4$.



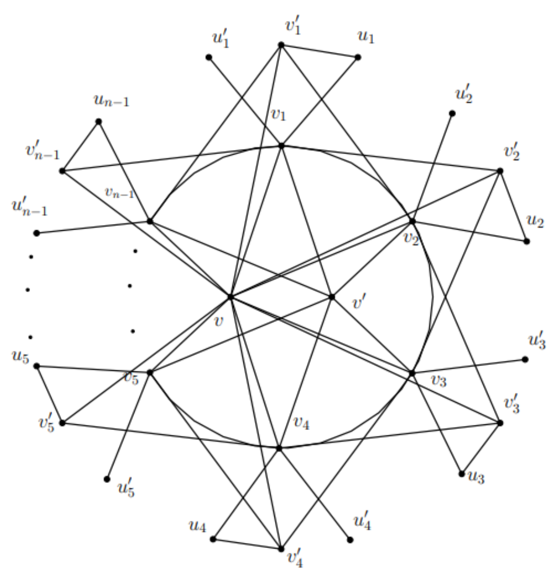
$S'(W_{1,n-1})$
Figure 2.4

Theorem 2.4 *For the Helm graph* $H_n (n \geq 4), \bar{\gamma}_d(S'(H_n)) = 2(n-1)$.

Proof. Let H_n be the helm graph obtained from a wheel graph $W_{1,n-1}$ by attaching a pendant edge for each rim vertices $v_1, v_2, \cdots, v_{n-1}$. Let the pendant vertices be $u_1, u_2, \cdots, u_{n-1}$ and v be the apex vertex of $W_{1,n-1}$. Let $v_{1'}, v_{2'}, \cdots, v_{n-1}{}'$ be the duplicated vertices of $v_1, v_2, \cdots, v_{n-1}$, $u_{1'}, u_{2'}, \cdots, u_{n-1}{}'$ be the duplicated vertices of $u_1, u_2, \cdots, u_{n-1}$ respectively and v' be the duplicated vertex of v, shown in figure 2.5. Then $V(S'(H_n)) = \{v, v_1, v_2, \cdots, v_{n-1}, u_1, u_2, \cdots, u_{n-1}, v_{1'}, v_{2'}, \cdots, v_{n-1}{}', u_{1'}, u_{2'}, \cdots, u_{n-1}{}', v'\}$ and $|V(S'(H_n))| = 4n - 2$.



H_n



$S'(H_n)$

Figure 2.5

Let $S = \{u_{1'}, u_{2'}, \cdots, u_{n-1}'\}$ be the set of all end vertices of $S'(H_n)$. By Theorem 1.3, $\bar{\gamma}_d(S'(H_n)) \geq n-1$. Since $I_D[u_{1'}, u_{2'}] = V(S'(H_n))$, S is a detour set of $S'(H_n)$. Also, we can see that degree of each $v_i, 1 \leq i \leq n-1$ is 8 and hence v_i dominates the most number of vertices in $S'(H_n)$. Hence, consider the set $S_1 = S \cup \{v_1, v_2, \cdots, v_{n-1}\}$ which dominates every vertices in $S'(H_n)$. Clearly, S_1 is a detour dominating set of $S'(H_n)$. Now, consider the complement graph $\overline{S'(H_n)}$ where $u_{1'}$ is adjacent to every vertices other than v_1 and $u_{2'}$ is adjacent to every vertices other than v_2. Clearly, $u_{1'}$ and $u_{2'}$ dominates every vertices in $\overline{S'(H_n)}$. Since $u_{1'}$ and $u_{2'}$ are in S_1, $S_1 = \{u_{1'}, u_{2'}, \cdots, u_{n-1}', v_1, v_2, \cdots, v_{n-1}\}$ is a minimum detour global dominating set of $S'(H_n)$ and so $\bar{\gamma}_d(S'(H_n)) \leq |S_1| = 2(n-1)$. Moreover, if we delete any vertex from S_1, then S_1 will never be a detour global dominating set of $S'(H_n)$. Thus we conclude that S_1 is a minimum detour global dominating set of $S'(H_n)$. Hence, $\bar{\gamma}_d(S'(H_n)) = 2(n-1)$.

Corollary 2.5 *For the Helm graph* $H_n (n \geq 4), \bar{\gamma}_d(S'(H_n)) = 2(\bar{\gamma}_d(H_n) - 1)$.
Proof. Clearly, H_n contains $2n-1$ vertices in which $n-1$ vertices are end vertices. Therefore, $\bar{\gamma}_d(H_n) \geq n-1$. It is clear that $n-1$ end vertices and the central vertex together form a detour global dominating set of H_n. Hence, $\bar{\gamma}_d(H_n) = n$. By Theorem 2.4, $\bar{\gamma}_d(S'(H_n)) = 2(n-1) = 2(\bar{\gamma}_d(H_n) - 1)$.

References

[1] F.Buckley and F.Harary, Distance in Graphs, *Addison-Wesley, Redwood City,*(1990).
[2] G. Chartrand, N. Johns and P. Zang, Detour Number of a graph, *Util. Math.***64**, 97-113 (2003).
[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, *Marcel Dekker,Inc.*, New York, (1998).
[4] C. Jayasekaran and S.V. Ashwin Prakash, Detour global domination number of some graphs *Malaya Journal of Matematik*, **S(1)** 552-555 (2020). **https://doi.org/10.26637/MJM0S20/0105**
[5] C. Jayasekaran and S.V. Ashwin Prakash, Detour global domination for splitting graphs, *Discrete Mathematics, Algorithms and Applications*, (2022).**https://doi.org/10.1142/S1793830922501221**
[6] E. Sampath Kumar, The Global Domination Number of a Graph, *Journal of Mathematical and Physical Sciences*, **23(5)** 377-385 (1989).
[7] E. Sampath Kumar and H.B. Walikar, On the Splitting Graph of a Graph *The Karnataka University Journal*, **XXV** and **XXVI**(Combined) (1980-1981).
[8] D.B. West, Introduction to Graph Theory, *Second Ed., Prentice-Hall, Upper Saddle River, NJ,*(2001).